On the Effect of Friend Feedbacks

Albert Compta, Josep Ferrer and Marta Peña

Departament de Matemàtica Aplicada I, ETSEIB, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain

Abstract. Given $\mathscr{S}$ an (A,B)-invariant subspace, we prove that the set of friend feedbacks is a $(nm-md+dq)$-dimensional linear variety, which can be considered as the direct sum of the feedbacks of the restriction to $\mathscr{S}$ and the co-restriction to $\mathscr{S}^{\perp}$. In particular, if (A,B) is controllable and $\mathscr{S}$ is a controllability subspace, both pole assignments are simultaneously possible by means of a convenient friend feedback.

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INTRODUCTION

Given a finite-dimensional time invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

$A \in M_n(\mathbb{R})$, $B \in M_{n,m}(\mathbb{R})$, $C \in M_{r,n}(\mathbb{R})$, the notions of (A,B)-invariant and (C,A)-invariant subspaces, or controlled and conditioned invariant subspaces, respectively, were introduced in [1]. They play a fundamental role in geometric control theory (see, for example, [2]). We recall that a subspace $\mathscr{S}$ is (A,B)-invariant if $A\mathscr{S} \subset \mathscr{S} + \operatorname{Im} B$ and that the map $\mathscr{S} \longrightarrow \mathscr{S}^{\perp}$ (the orthogonal to $\mathscr{S}$) is a bijection between the set of (A,B) and (B^t,A^t)-invariant subspaces.

In [2] one asks for the effects of feedbacks beyond the well-known use of shifting poles. For example, if the pair (A,B) is controllable, it is clear that a subfamily of feedbacks give the miniversal deformation of A in [3], that is to say, any Jordan form near the original one of A can be obtained by little feedbacks of this subfamily. It can be shown that the remainder feedbacks have no effect on the Jordan invariants of A because they are in fact a conjugation.

Here we focus in the subfamily of friend feedbacks with regard to an (A,B)-invariant subspace $\mathscr{S}$, that is, the feedbacks F such that $(A+BF)\mathscr{S} \subset \mathscr{S}$. For example, they appear in the Disturbance Decoupling Problem, when $\mathscr{S}$ is chosen the maximal (A,B)-invariant subspace contained in $\operatorname{Ker} C$. Our first step (Theorem 3) is showing that the set of friend feedbacks is a $(nm-md+dq)$-dimensional linear variety, where $d=\dim\mathscr{S}$ and $q=\dim(\mathscr{S}\cap\operatorname{Im} B)$.

The key point is that any friend feedback induces a feedback in the restriction of (A,B) to $\mathscr{S}$ and in the so-called co-restriction of the dual pair (B^t,A^t) to the conditioned invariant subspace $\mathscr{S}^{\perp}$. The main result (Theorem 9) asserts that any both prescribed feedbacks on the restriction and the co-restriction can be induced simultaneously by a friend feedback, which is uniquely determined by these requirements. In particular, if (A,B) is controllable and $\mathscr{S}$ is a controllability subspace, one can choose a friend feedback in order to obtain simultaneously prescribed pole assignment both in the restriction and the co-restriction (Corollary 11).

We make use of the following notation. We denote by $\mathbb{R}$ the field of real numbers. We write $M_{n,m}(\mathbb{R})$ for the vector space of matrices with n rows and m columns with entries in $\mathbb{R}$. If $n=m$ we write simply $M_n(\mathbb{R})$. If M is a matrix we denote by M^t its transpose. If $M \in M_{n,m}(\mathbb{R})$ we identify M with the linear map $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ defined in a natural way.

Throughout the paper, we consider $\mathscr{S} \subset \mathbb{R}^n$ a subspace, $\mathscr{S}^{\perp}$ its orthogonal, $d=\dim\mathscr{S}$ and $X, X^{\perp}$ matrices of a basis of $\mathscr{S}, \mathscr{S}^{\perp}$, respectively. We will consider pairs of matrices $(A,B) \in M_n(\mathbb{R}) \times M_{n,m}(\mathbb{R})$ and the block-equivalence between pairs of matrices, known as Brunovsky Kronecker (or BK)-equivalence. We will assume without loss of generality that B has full column rank m. Let $q=\dim(\mathscr{S}\cap\operatorname{Im} B)$.

SET-UP

Exercise 5.6 in [2] considers the pair (A,B) and the (A,B)-invariant subspace $\mathscr{S}$:

$$A=\begin{pmatrix}1&0&0&0&1&0\\0&0&1&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&1\end{pmatrix},\quad B=\begin{pmatrix}0&1&0\\0&0&0\\1&0&0\\0&0&1\\0&0&0\\0&0&0\end{pmatrix},\quad \mathscr{S}=\begin{pmatrix}1&0&0&1&2\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

One computes the set of the so-called "friend feedbacks", that is to say, the matrices F such that $(A+BF)\mathscr{S}\subset\mathscr{S}$

$$F=\begin{pmatrix}f_{11}&f_{12}&f_{13}&f_{14}&f_{15}&f_{16}\\f_{21}&f_{22}&f_{23}&f_{24}&f_{25}&f_{26}\\0&0&0&f_{34}&-1&0\end{pmatrix},$$

and the corresponding restrictions of $A+BF$ to $\mathscr{S}$

$$(A+BF)|_{\mathscr{S}}=\left(\begin{array}{ccc|cc}1+f_{21}&f_{22}&f_{23}&2+f_{21}+f_{25}&2f_{21}+f_{26}\\0&0&1&0&0\\f_{11}&-1+f_{12}&f_{13}&f_{11}+f_{15}&2f_{11}+f_{16}\\\hline 0&0&0&0&0\\0&0&0&0&1\end{array}\right),\qquad (*)$$

where the upper left block is the restriction to the supremal controllability subspace in $\mathscr{S}$ and, hence, the bottom right block is the quotient map. One pointed out that the spectrum of the first one is arbitrarily assignable by means of a suitable friend feedback, whereas no change is possible in the second spectrum. However, other possible effects of the parameters in F, mainly f_{14}, f_{24} and f_{34}, are not obvious.

Our approach is based on the "restriction" $(\bar{A},\bar{B})$ of (A,B) to $\mathscr{S}$, and in the co-restriction (B^c,A^c) to $\mathscr{S}^\perp$. We will see that $(*)$ is the class of matrices $\bar{A}+\bar{B}\bar{F}$, and that the remainder parameters f_{14}, f_{24} and f_{34} define just the feedbacks F^c of the co-restriction (B^c,A^c). In general, we will see that there is a bijection between the friend feedbacks F and the pairs $(\bar{F},F^c)$ of general feedbacks for the restriction and the co-restriction pairs.

FRIEND FEEDBACKS

We recall the basic definitions we will deal with (see, for example, [4]):

Definition 1 *Given a pair (A,B) and a subspace $\mathscr{S}\subset\mathbb{R}^n$:*

*(1) $\mathscr{S}\subset\mathbb{R}^n$ is an (A,B)-*invariant subspace *(or* controlled invariant subspace*) if there exists $F\in M_{m,n}(\mathbb{R})$ such that*

$$(A+BF)\mathscr{S}\subset\mathscr{S}.$$

(2) This is equivalent to

$$(A^t+F^tB^t)\mathscr{S}^\perp\subset\mathscr{S}^\perp$$

*saying that $\mathscr{S}^\perp\subset\mathbb{R}^n$ is a (B^t,A^t)-*invariant subspace *(or* conditioned invariant subspace*).*

(3) The matrices $F\in M_{m,n}(\mathbb{R})$ verifying (1)-(2) are called friend feedbacks *and we will denote the set of them by $\mathscr{F}$.*

Remark 2 *In our approach, it is convenient to reformulate the above condition (1) in matricial terms as follows: there exist $F\in M_{m,n}(\mathbb{R})$ and $R\in M_d(\mathbb{R})$ such that*

$$(A+BF)X=XR.$$

Our first goal is the geometric structure of the set of friend feedbacks.

Theorem 3 *The set $\mathscr{F} \subset M_{m,n}(\mathbb{R})$ of friend feedbacks is a linear subvariety*

$$\mathscr{F} = F_0 + \mathscr{F}_0,$$

where $F_0 \in \mathscr{F}$ is any fixed one and $\mathscr{F}_0$ a linear subspace having dimension

$$\dim \mathscr{F}_0 = m(n-d) + dq.$$

Example 4 *For the example considered in the Set-up, we have:*

$$F_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathscr{S} \cap \operatorname{Im} B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \dim \mathscr{F}_0 = 3(6-5) + 5 \cdot 2 = 13.$$

RESTRICTION AND CO-RESTRICTION

The effects of the friend feedbacks will be reflected in the "restriction" of (A,B) to $\mathscr{S}$ (see, for example, [5], [6]) and the "co-restriction" of (B^t, A^t) to $\mathscr{S}^\perp$, which we define in a natural way:

Definition 5 *Let $\mathscr{S}$ be an (A,B)-invariant subspace.*

(1) We call restriction *of (A,B) to $\mathscr{S}$ the set of pairs $\{(\bar{A},\bar{B})\} \subset M_d(\mathbb{R}) \times M_{d,q}(\mathbb{R})$ obtained as follows when varying $F \in \mathscr{F}$ and the bases in $\mathscr{S}$ and in $\mathscr{S} \cap \operatorname{Im} B$:*
For a fixed basis X of $\mathscr{S}$ and a fixed friend feedback F, $\bar{A}_{XF}$ or simply $\bar{A}$ means the matrix of $A+BF$ restricted to $\mathscr{S}$ in the basis X or, equivalently,

$$(A+BF)X = X\bar{A}.$$

In particular, we write $\bar{A}_0$ the one corresponding to the fixed $F_0 \in \mathscr{F}$.
The matrices $\bar{B}$ are defined simply by:

$$X\bar{B} \text{ is a basis of } \mathscr{S} \cap \operatorname{Im} B.$$

We will refer to each of the above pairs $(\bar{A},\bar{B})$ as a restriction of (A,B) to $\mathscr{S}$.

(2) We call the co-restriction *of (A,B) to $\mathscr{S}^\perp$ the set of pairs $\{(B^c, A^c)\} \subset M_{m,n-d}(\mathbb{R}) \times M_{n-d}(\mathbb{R})$ obtained as follows when varying $F \in \mathscr{F}$ and the basis in $\mathscr{S}^\perp$:*
For a fixed basis $X^\perp$ of $\mathscr{S}^\perp$ and a fixed friend feedback F, $A^c_{X^\perp F^t}$ or simply A^c means the matrix of $A^t + F^tB^t$ restricted to $\mathscr{S}^\perp$ in the basis $X^\perp$ or, equivalently,

$$(A^t + F^tB^t)X^\perp = X^\perp A^c.$$

In particular, we write A^c_0 the one corresponding to the fixed $F_0 \in \mathscr{F}$.
Analogously, B^c means the matrix of B^t in the basis $X^\perp$ or, equivalently,

$$B^c = B^tX^\perp.$$

We will refer to each of the above pairs (B^c, A^c) as a co-restriction of (A,B) to $\mathscr{S}^\perp$.

Remark 6 *(1) Remark 2 warrants the above definition of $\bar{A}$. Analogously for A^c.*

(2) Clearly, $\bar{B}$ has maximal rank, but not, in general, B^c: $\quad \operatorname{rank}\bar{B} = q, \quad \operatorname{rank}B^c = \operatorname{rank}B - \operatorname{rank}\bar{B} = m - q.$

(3) Clearly, $X\bar{B} = BH$ for some $H \in M_{m,q}(\mathbb{R})$.

(4) Then the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}^m & \overset{(A,B)}{\longrightarrow} & \mathbb{R}^n \\ \begin{pmatrix} X & 0 \\ FX & H \end{pmatrix} \uparrow & & \uparrow X \\ \mathbb{R}^d \times \mathbb{R}^q & \overset{(\bar{A},\bar{B})}{\longrightarrow} & \mathbb{R}^d \end{array}$$

shows that the above definition is just the one in [6].

Example 7 *For the above example, we have:*

$$(1)\ \bar{A}_0 = \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$(2)\ X^{\perp} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad A_0^c = \begin{pmatrix} 0 \end{pmatrix}, \quad B^c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Indeed, the restrictions $(\bar{A},\bar{B})$ and the co-restrictions (B^c,A^c) are unique up to BK-equivalence:

Proposition 8 *With the notations in Definition 5:*

(1) The restriction $\{(\bar{A},\bar{B})\}$ is a BK-class in the set of linear maps $\mathscr{S} \times \mathbb{R}^m \longrightarrow \mathscr{S}$.
(2) The co-restriction $\{(B^c,A^c)\}$ is a BK-class in the set of linear maps $\mathscr{S}^{\perp} \longrightarrow \mathscr{S}^{\perp} \times \mathbb{R}^m$.

Now we can state our main theorem: there is a bijection between the friend feedbacks $\mathscr{F}$ and the pairs $(\bar{F},F^c)$ of general feedbacks for the restriction and the co-restriction pairs:

Theorem 9 *Given $\mathscr{S}$ an (A,B)-invariant subspace, let $(\bar{A}_0,\bar{B})$ be a restriction and (B^c,A_0^c) a co-restriction with regard to fixed bases of $\mathscr{S}$, $\mathscr{S}^{\perp}$ and $\mathbb{R}^m$, and a fixed friend feedback $F_0 \in \mathscr{F}$. Then, there are a direct decomposition of $\mathscr{F}_0$*

$$\mathscr{F} = F_0 + (\bar{\mathscr{F}}_0 \oplus \mathscr{F}_0^c),$$

and isomorphisms $\quad \varphi : M_{q,d}(\mathbb{R}) \longrightarrow \bar{\mathscr{F}}_0, \quad \psi : M_{m,n-d}(\mathbb{R}) \longrightarrow \mathscr{F}_0^c,$
such that $\quad (A + B(F_0 + \varphi(\bar{F}))X = X(\bar{A}_0 + \bar{B}\bar{F}), \quad (A^t + (F_0 + \psi(F^c))^t B^t)X^{\perp} = X^{\perp}(A_0^c + F^c B^c).$

That is to say: any friend feedback F of $\mathscr{S}$ induces feedbacks of $(\bar{A}_0,\bar{B})$ and (B^c,A_0^c); and conversely, for $\bar{F}$ and F^c prescribed feedbacks of $(\bar{A}_0,\bar{B})$ and (B^c,A_0^c), respectively, there is a unique friend feedback which induces $\bar{F}$ and F^c.

Example 10 *For the above example, we have that a friend feedback induces:*

$$\bar{A}_0 + \begin{pmatrix} f_{21} & f_{22} & f_{23} & f_{21}+f_{25} & 2f_{21}+f_{26} \\ f_{11} & f_{12} & f_{13} & f_{11}+f_{15} & 2f_{11}+f_{16} \end{pmatrix} \bar{B}, \qquad A_0^c + \begin{pmatrix} f_{24} & f_{14} & f_{34} \end{pmatrix} B^c.$$

As an application, simultaneous pole assignments in $(\bar{A},\bar{B})$ and (B^t,A^t) can be attempted by a friend feedback when natural controllability hypotheses hold.

Corollary 11 *In the conditions of Theorem 9, assume that (A,B) is controllable and $\mathscr{S}$ is a controllability subspace. Then there exists a friend feedback such that the spectrum of $\bar{A}$ and A^c can be shifted to prescribed ones by means of the induced feedbacks.*

REFERENCES

1. G. Basile and G. Marro, Controlled and conditioned invariant subspaces in linear system theory, *J. Optim. Theory Appl.*, **3**, 306–315, (1969).
2. W.M. Wonham, *Linear Multivariable Control: A Geometric Approach*, Springer, New York, (1979).
3. V.I. Arnold, On Matrices Depending on Parameters, *Uspekhi Mat. Nauk.*, **26**, 101–114 (1971).
4. I. Gohberg, P. Lancaster and L. Rodman, *Invariant Subspaces of Matrices with Applications*, Wiley, New York, (1986).
5. I. Baragaña and I. Zaballa, Block similarity invariants of restrictions to (A,B)-invariant subspaces, *Linear Algebra Appl.*, **220**, 31–62 (1995).
6. F. Puerta, X. Puerta and I. Zaballa, On the parametrization of the controllability subspaces of a controllable pair, *Linear Algebra Appl.*, **401**, 429–444 (2005).
7. F. Puerta and X. Puerta, On the geometry of the set of controllability subspaces of a pair (A,B), *Linear Algebra Appl.*, **351–352**, 585–599 (2002).